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Indian Statistical Institute
B.Math.(Hons.) II Year
First Semester Back Paper Examination, 2005-2006
Algebra III

Time: 3 hrs | Date: -12-05 | Total Marks : 50

Attempt Qn.1 and any 3 of the other 4 questions (Qn. 2-5)

1. State whether the following statements are true or false and justify your answer (quote a relevent theorem or give an argument/counterexample)

   a) Let $K = Q(\alpha,\ \beta)$ where $\alpha = \sqrt[4]{5}, \beta = \sqrt[5]{6}$. Then $[K : Q] = 20$.

   b) The Galois group of $x^3 - 3x + 1$ over Q is S_3.

   c) Let $F \subseteq L \subseteq K$ be fields with K/F Galois. Then L/F is also Galois.

   d) $K = \mathbb{F}_2[\mathrm{x}]/(\mathrm{x}^3 + \mathrm{x} + 1)$ is a field. (4 × 5)

2. a) What is the irreducible polynomial of $\sqrt{3} + \sqrt{5}$ over $Q(\sqrt{15})$?

   b) Prove that the polynomial $f(x) = x^3 - 3x + 4$ is irreducible over Q. Let α be a root of $f(x)$. Find the inverse of $(\alpha^2 + \alpha + 1)$ in $Q(\alpha)$ explicitly in the form $a + b\alpha + c\alpha^2$ where $a, b, c, \in Q$. (5+5)

3. a) State the Main Theorem of Galois Theory.

   b) Let K be the splitting field of $x^3 - 2$ over Q. Write the complete list of intermediate fields of K/Q and the corresponding subgroups of $G(K/Q)$. Which subextensions of K/Q are Galois? (5+5)

4. Let $s = e^{2\pi i/5}$ and let $K = Q(s)$. Prove that K is a splitting field of $x^5 - 1$ over Q and determine the degree $[K : Q]$. Given an explicit description of the elements of the Galois group $G(K/Q)$ in terms of their action on s. (10)

5. a) What is the discriminant of $x^4 + 1$? What is the Galois group of $x^4 + 1$ over Q.

   b) Suppose that a real quantic polynomial has a positive discriminant. Prove that it cannot have exactly two real roots. (5+5)